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LAMINAR MIXING LAYER ON THE BOUNDARY OF TWO FLOWS IN THE PRESENCE OF A LONGITUDINAL PRESSURE GRADIENT

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We consider, in a linear formulation, the problem concerning the laminar mixing layer on the boundary of two flows of an incompressible liquid with a small difference in their Bernoulli constants; we assume the presence of longitudinal pressure gradient. We determine the velocity distribution in the mixing layer, the magnitude of the displacement thickness and the momentum loss thickness. For the case in which there is no longitudinal pressure gradient we calculate the force effect of the one flow on the other.

1. As the Reynolds number $R \to \infty$, it follows, from an analysis of the possible limiting flows of a viscous liquid with stationary discontinuity zones (see [1]), that the discontinuity Δ in the Bernoulli constant at the boundary of the discontinuity zone tends to zero as $R \to \infty$ ($\Delta = (u_1^2 - u_2^2) / u_\infty^2$, where u_1 and u_2 are the velocities on the outer and inner sides, respectively, at the boundary points, and u_∞ is the velocity of the unperturbed flow) and acquires a value calculated from the parameters of the mixing layer when $\Delta \ll 1$ (Fig. 1, a and b).

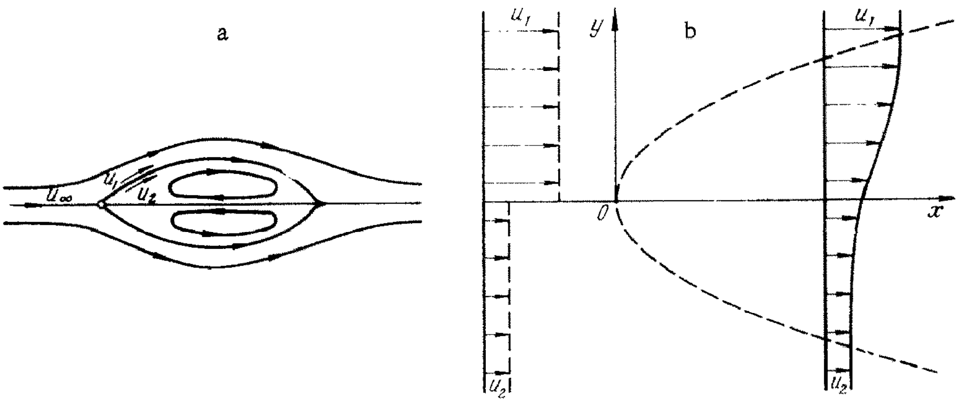

Fig. 1

For large Reynolds numbers the laminar mixing layer on the boundary of two planar flows of an incompressible liquid is described by the Prandtl boundary layer equations, subject to the boundary and initial conditions

$$u(x) = u_1(x),\ y = \infty;\ u(x) = u_2(x),\ y = -\infty;\ u(0, y) = \varphi(y) \qquad (1.1)$$

Here, and in the sequel, u and v are the components of the velocity in the directions of the x and y axes of a Cartesian coordinate system, and $\varphi(y)$ is the distribution of the velocity component u for $x = 0$. Within the limits of boundary layer theory the pressure across the mixing layer is constant, so that according to Bernoulli equation we have the following condition to be imposed on the functions $u_1(x)$ and $u_2(x)$: $\Delta = \text{const}$ along the whole mixing boundary.

In the case of a boundary layer near a rigid wall the longitudinal velocity component always changes by an amount equal to the velocity at the outer boundary of the layer; for this reason linearization of the equations and boundary conditions for this component is not possible. On the other hand, in the case of the mixing layer, when the difference of the velocities $u_1(x) - u_2(x)$ is small in comparison with the velocity itself (or, equivalently, when $\Delta \ll 1$), linearization of the equations and boundary condititions is possible. To linearize the problem we write the velocity in the mixing layer in the form $u(x, y) = u_1(x) + \varepsilon(x, y)$ (here $\varepsilon(x, y)$ is a small addition to the velocity on the outer boundary of the mixing layer). Assuming that $v(x, 0) = 0$, it follows from the continuity condition for an incompressible liquid that

$$v(x, y) = -y\frac{du_1}{dx} - \int_0^y \frac{\partial \varepsilon}{\partial x}\, dy \qquad (1.2)$$

($y = 0$ is the separating stream line). Substituting Eq. (1.2) into the equation of the boundary layer and replacing u by $u_1(x) + \varepsilon(x, y)$, we obtain, upon discarding the product of small quantities, the following linear equation for $\varepsilon(x, y)$:

$$\frac{\partial(\varepsilon u_1)}{\partial x} - \frac{y}{u_1}\frac{du_1}{dx}\frac{\partial(\varepsilon u_1)}{\partial y} = \frac{\nu}{u_1}\frac{\partial^2(\varepsilon u_1)}{\partial y^2} \qquad (1.3)$$

We now transform the boundary conditions (1.1), written for the velocity component u, to a form corresponding to the new unknown quantity $\varepsilon(x, y)$: $\varepsilon = 0$ and $\varepsilon u_1 = 0$ for $y = \infty$. For $y = -\infty$ the velocities $u_1(x)$ and $u_2(x)$ are related by the condition of equality of the pressure on the upper and lower boundaries of the mixing layer (Fig. 1 b); therefore, using the Bernoulli equations for the upper and lower flows and neglecting the term involving $\varepsilon^2(x, -\infty)$, we have

$$\varepsilon(x, -\infty)\, u_1 = {}^1/_2 u_\infty{}^2 \Delta = \text{const}$$

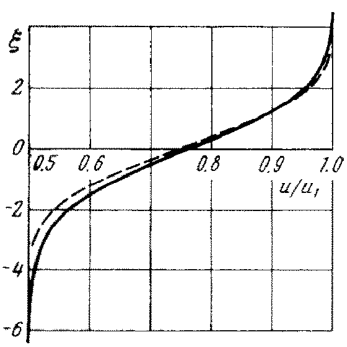

Fig. 2

Thus, if the difference $u_1(x) - u_2(x)$ is small, the problem concerning the determination of the velocity in the mixing layer, in a linear formulation, reduces to the solution of Eq. (1.3), subject to the boundary and initial conditions

$$\varepsilon u_1 = 0, \quad y = \infty; \quad \varepsilon u_1 = {}^1/_2 u_\infty{}^2 \Delta, \quad y = -\infty; \qquad (1.4)$$

$$\varepsilon(0, y) = \varphi(y) - u_1(0)$$

2. For the solution of the problem formulated above we place the coordinate origin at the beginning of the mixing layer and introduce the independent variable

$$\xi = y\sqrt{u_\infty / [\nu f(x)]}$$

($f(x)$ is a function which will be defined later). Replacing the partial derivatives in Eq. (1.3) by derivatives with respect to ξ, we obtain the equation for εu_1 and the corresponding boundary conditions in the form

$$\frac{d^2(\varepsilon u_1)}{d\xi^2} \Big/ \xi \frac{d(\varepsilon u_1)}{d\xi} = -\frac{u_1 f(x)}{u_\infty} \frac{d}{dx} \ln [u_1 f^{1/2}(x)] \tag{2.1}$$

$$\varepsilon u_1 = 0, \quad \xi = \infty; \quad \varepsilon u_1 = {}^1/_2 u_\infty{}^2 \Delta, \quad \xi = -\infty \tag{2.2}$$

In order that the solution of Eq. (2.1) for εu_1 may depend only on ξ, it is necessary that the left and righ sides of Eq. (2.1) be equal to a certain constant c. The solution of the equation corresponding to the left side of (2.1) with the boundary conditions (2.2) exists only for $c < 0$ and has the form

$$\varepsilon u_1 = -\frac{u_\infty{}^2 \Delta}{2\sqrt{2\pi}} \int\limits_{\xi\sqrt{-c}}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt \tag{2.3}$$

As the solution of the equation corresponding to the right side of (2.1) is the function

$$f(x) = -2c \frac{u_\infty}{u_1{}^2} \int\limits_0^x u_1\, dx + c_1/u_1^2 \tag{2.4}$$

Having in mind a comparison of our results with the numerical solutions given in [2] of the complete boundary layer equations for the laminar mixing layer in the absence of a longitudinal pressure gradient, we choose the constants c and c_1 so that in the absence of a pressure gradient $(u_1 = u_\infty)$ the variable ξ will coincide with the dimensionless variable η used in [2]. Thus, it is necessary to take $c = -{}^1/_2$ and $c_1 = 0$. We note that in Eq. (2.3) the quantity $\xi\sqrt{-c}$ does not depend on an actual value for c ; hence, neither does the quantity εu_1 .

Using the equation obtained above for the velocity distribution, we calculate the displacement thickness $\delta_1{}^*$ and the momentum loss thickness $\delta_2{}^*$

$$\delta_1{}^* = \frac{1}{u_1}\int\limits_0^\infty (u_1 - u)\, dy = \frac{u_\infty{}^2\Delta}{2\sqrt{2\pi}\, u_1{}^2}\int\limits_0^x \int\limits_{\xi\sqrt{2}}^\infty \exp\left(-\frac{t^2}{2}\right) dt\, dy =$$

$$\frac{f(x)\, u_\infty{}^2\Delta}{2\sqrt{\pi}\, u_1{}^2 R_f^{1/2}}, \qquad R_f = \frac{u_\infty f(x)}{\nu}$$

$$\delta_1{}^{**} = \int\limits_0^\infty \frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) dy = \int\limits_0^\infty \left(1 + \frac{\varepsilon}{u_1}\right) \frac{\varepsilon}{u_1}\, dy$$

Neglecting the term involving ε^2, we find, within the scope of the linear theory, that

$$\delta_1{}^* = \delta_1{}^{**} = \delta_2{}^* = \delta_2{}^{**}$$

$$\left(\delta_2{}^* = \frac{1}{u_2}\int\limits_0^{-\infty} (u_2 - u)\, dy, \qquad \delta_2{}^{**} = \frac{1}{u_2{}^2}\int\limits_0^{-\infty} u\,(u_2 - u)\, dy\right)$$

3. In the absence of a longitudinal pressure gradient $(u_1 = u_\infty)$ we have $f(x) = x$, $\xi = y\sqrt{u_\infty / (\nu x)}$, and

$$\varepsilon(x, y) = -\frac{u_\infty \Delta}{2\sqrt{2\pi}} \int\limits_{\xi\sqrt{2}}^\infty \exp\left(-\frac{t^2}{2}\right) dt \tag{3.1}$$

In Fig. 2 we show the variation of the velocity in the mixing layer for $u_2 / u_1 = 0.5$, corresponding to the expression (3.1) (dashed curve); for comparison we also show the results, taken from [2], obtained in numerically integrating the complete boundary layer equations (solid curve). In the Table 1 we give values of the velocity on the separating stream line ($y = 0$), obtained from the expression (3.1) and from the numerical calculations in [2], for various values of u_2 / u_1. It is evident that the expression (3.1) describes the velocity distribution in the mixing layer quite satisfactorily.

Table 1

u_2/u_1	$u(x, 0)/u_1$	
	[2]	(3.1)
0	0.5873	0.500
0.5	0.7657	0.750
0.75	0.8784	0.875

For the case of a mixing layer without pressure gradient, we have

$$\frac{\delta_1^*}{x} = \frac{\Delta}{2\sqrt{2\pi}R_x^{1/2}}, \quad c_x = \frac{\Delta}{\sqrt{\pi}R_x^{1/2}}$$

Here c_x is the dimensionless coefficient of the force effect of one flow on the other. The corresponding quantities for the boundary layer on a plate are

$$\frac{\delta_1^*}{x} = \frac{1.72}{R_x^{1/2}}, \quad c_x = \frac{1.328}{R_x^{1/2}}$$

We note that the law of velocity distribution in the laminar mixing layer when the pressure gradient is absent (3.1) was established for the first time in [3]. It was obtained here as a particular case of the law for the mixing layer in the presence of the pressure gradient. A linearized solution of the boundary layer equation in the absence of longitudinal pressure gradient has also been used in the solution of other problems concerned with laminar liquid and gas flows (see, for example, [4]).

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